U. S. AIR FORCE

PROJECT RAND

RESEARCH MEMORANDUM

GAMES OF SURVIVAL

Melvin Hausner

RM-776

12 February 1952

Assigned to ____________________

This is a working paper. It may be expanded, modified, or withdrawn at any time. The views, conclusions, and recommendations expressed herein do not necessarily reflect the official views or policies of the United States Air Force.

The RAND Corporation
1700 MAIN ST. • SANTA MONICA • CALIFORNIA

Report Documentation Page

Form Approved
OMB No. 0704-0188

1. REPORT DATE	2. REPORT TYPE	3. DATES COVERED
12 FEB 1952		**00-00-1952 to 00-00-1952**

4. TITLE AND SUBTITLE
Games of Survival

5a. CONTRACT NUMBER

5b. GRANT NUMBER

5c. PROGRAM ELEMENT NUMBER

6. AUTHOR(S)

5d. PROJECT NUMBER

5e. TASK NUMBER

5f. WORK UNIT NUMBER

7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES)
Rand Corporation,Project Air Force,1776 Main Street, PO Box 2138,Santa Monica,CA,90407-2138

8. PERFORMING ORGANIZATION REPORT NUMBER

9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)

10. SPONSOR/MONITOR'S ACRONYM(S)

11. SPONSOR/MONITOR'S REPORT NUMBER(S)

12. DISTRIBUTION/AVAILABILITY STATEMENT
Approved for public release; distribution unlimited

13. SUPPLEMENTARY NOTES

14. ABSTRACT

15. SUBJECT TERMS

16. SECURITY CLASSIFICATION OF:			17. LIMITATION OF ABSTRACT	18. NUMBER OF PAGES	19a. NAME OF RESPONSIBLE PERSON
a. REPORT	b. ABSTRACT	c. THIS PAGE			
unclassified	**unclassified**	**unclassified**	**Same as Report (SAR)**	**10**	

Standard Form 298 (Rev. 8-98)
Prescribed by ANSI Std Z39-18

SUMMARY

A 2x2 zero-sum game is played continually by 2 people, the object being either survival or ruin of the opponent. There is a constant sum of money in the game. It is shown that in general the game has a solution and that these two motives are equivalent. Methods for calculating the solution are given and a particular example is worked out. A singular example is given with a discussion. A game is shown to be singular only if on a single play there is a possibility of a draw (there is a zero entry in the matrix).

GAMES OF SURVIVAL

Melvin Hausner

We consider the following "game": Players I and II play the two-person zero-sum game whose payoff matrix (to I) is

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}.$$

Each player starts with a fixed amount of money, and they play continually until one of the players runs out of money. Of course, the total amount n of money in the game is fixed.

Some general remarks are in order. In the game

$$A = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix},$$

player I can guarantee his survival by playing row 1. If he announces this strategy, player II plays column 2 and both players guarantee survival. But if I's object is to ruin II, he must eventually play row 2 with a positive probability (we assume that optional strategies are announced). He then subjects himself to the chance of ruin. Hence, we have two possible (and possibly conflicting) motives: survival and ruin of the opponent.

Furthermore, as in any two-person zero-sum game, we may eliminate dominated rows and dominating columns. This may be done for either objective. Therefore, we shall assume that A has no saddle-point. If we normalize by taking $a_{11} = \max a_{ij}$, then I loses if $a_{11} < 0$. If $a_{11} = 0$, and $a_{12} < 0$, II plays column 2 and always wins unless $a_{22} = 0$. But if $a_{11} = a_{22} = 0$, $a_{12} < 0$ the solution is clear if $a_{21} = 0$. But if $a_{21} < 0$, player II randomizes equally between columns 1 and 2 and he wins with probability 1. Hence, we may take $a_{11} > 0$. If $a_{12} > 0$, player I wins, so that we may take $a_{12} \leqq 0$. If $a_{21} > 0$, player II will play column 2 and the outcome is determined according as $\max (a_{21}, a_{22}) \gtreqless 0$. Hence, we may take $a_{21} \leqq 0$. Since no saddle-point occurs, we have $a_{22} > a_{12}, a_{21}$. We summarize as follows: $a_{11} > 0$; $a_{21}, a_{12} \leqq 0$; $a_{12}, a_{21} < a_{22} \leqq a_{11}$.

We now consider the game where I wants to maximize his probability of survival and II wants to minimize I's probability of survival. A mixed strategy for I is a sequence $(p_1, \ldots, p_{n-1})$ of probabilities. p_k is the probability of I playing row 1 when he has k units of money. (We assume here that a player's strategy is dictated by his money--not the past history of the game.) In a pure strategy $p_k = 1$ or 0. A similar definition of a strategy $(r_1, \ldots, r_{n-1})$ exists for player II. r_k is the probability of playing column 1 when <u>player I</u> has k units of money. We shall show that under certain conditions the game has a value and associated mixed strategies.

First consider the minorant version of the game: I states his mixed strategy. II then chooses his pure strategy (he has 2^{n-1} such strategies), which minimizes I's probability of survival, giving $f(p_1, \ldots, p_{n-1})$. We assume that <u>f is continuous</u>, and that the maximum is attained. (Later, we show that this is not necessarily true.) Player I then uses the strategy $(p_1, \ldots, p_{n-1})$ and is assured of the probability $v(k)$ of survival if he has k units of money. Player II can restrict this probability to exactly $v(k)$. We have

$$(1) \qquad \begin{cases} v(r) = 0, & r \leqq 0 , \\ v(r) = 1, & r \geqq n , \end{cases}$$

and

$$v(k) = \max_{0 \leqq p \leqq 1} \ \min \begin{Bmatrix} pv(a_{11} + k) + (1 - p)\, v(a_{21} + k), \\ pv(a_{12} + k) + (1 - p)\, v(a_{22} + k) \end{Bmatrix} .$$

Clearly $v(k)$ is monotonic increasing with k. Since A has no saddle-point, max min occurs when

$$(2) \quad pv(a_{11} + k) + (1-p)\, v(a_{21} + k) = pv(a_{12} + k) + (1-p)\, v(a_{22} + k).$$

Hence,

$$(3) \quad p_k = \frac{v(a_{22} + k) - v(a_{21} + k)}{v(a_{22} + k) + v(a_{11} + k) - v(a_{12} + k) - v(a_{21} + k)} ,$$

and

$$(4) \quad v(k) = \frac{v(a_{22} + k)\, v(a_{11} + k) - v(a_{12} + k)\, v(a_{21} + k)}{v(a_{22} + k) + v(a_{11} + k) - v(a_{12} + k) - v(a_{21} + k)} .$$

Assuming that a_{ij} is an integer, these equations form $n-1$ equations in $n-1$ "unknowns" $v(k)$. If the denominator is 0, then p_k is arbitrary and the numerators involved are 0.

It is important to observe that the probability $v(k)$ of I's survival is independent of II's strategy if I plays optimally.

In the same way, we consider the majorant game, where II must state his mixed strategy and I chooses his strategy, afterwards. Letting $V(k)$ be the probability of I's survival if both play optimally, I having k units, we have

$$(1)' \quad \begin{cases} V(r) = 0, & r \leqq 0 \\ V(r) = 1, & r \geqq n \end{cases},$$

and

$$V(k) = \min_r \quad \max \begin{cases} rV(a_{11} + k) + (1-r)\, V(a_{12} + k), \\ rV(a_{21} + k) + (1-r)\, V(a_{22} + k). \end{cases}$$

As before, min max occurs when

$$(2)' \quad rV(a_{11} + k) + (1-r)\, V(a_{12} + k) = rV(a_{21} + k) + (1-r)\, V(a_{22} + k),$$

so that

$$(3)' \quad r_k = \frac{V(a_{22} + k) - V(a_{12} + k)}{V(a_{22} + k) + V(a_{11} + k) - V(a_{12} + k) - V(a_{21} + k)} .$$

Again, the probability of I's non-ruin is independent of I's strategy, provided II plays optimally. We have assumed that the analogous $F(r_1, \ldots , r_{n-1})$ which II minimizes is continuous.

We now prove that the game has a value. Suppose that I plays his strategy (3) of the minorant game and II plays his strategy (3)' of the majorant game. Since the payoff for (3) is independent of II's play, I's probability of survival is $v(k)$. But the payoff for (3)' is independent of I's play and hence I's probability of survival is $V(k)$. Hence, $V(k) = v(k)$. The game has a solution: I can guarantee himself a probability

$\geqq v(k)$ of survival and II can guarantee a probability $\leqq v(k)$ of I's survival.

Both optimal strategies are given in terms of $v(k)$ which was defined intrinsically. In practice, the system of equations (4) will uniquely define $v(k)$ subject to the conditions (1) and $0 \leqq v(k) \leqq 1$ and $v(k+1) \geqq v(k)$. Of course, similar results occur when the play is for II's survival.

We offer an example for $n = 4$. The play is for I's survival:

$$A = \begin{pmatrix} 2 & -1 \\ -2 & 1 \end{pmatrix}, \quad n = 4.$$

From (4), we obtain

$$v(1) = \frac{v(3)\ v(2) - v(0)\ v(-1)}{v(3) + v(2) - v(0) - v(-1)} = \frac{v(3)\ v(2)}{v(3) + v(2)},$$

$$v(2) = \frac{v(4)\ v(3) - v(1)\ v(0)}{v(4) + v(3) - v(1) - v(0)} = \frac{v(3)}{1 + v(3) - v(1)},$$

$$v(3) = \frac{v(5)\ v(4) - v(2)\ v(1)}{v(5) + v(4) - v(2) - v(1)} = \frac{1 - v(2)\ v(1)}{2 - v(2) - v(1)}.$$

Observe that the denominators obviously do not vanish. These equations may be solved and have one solution:

$$v(1) = 1 - \frac{\sqrt{2}}{2}, \quad v(2) = \frac{1}{2}, \quad v(3) = \frac{\sqrt{2}}{2}.$$

Substituting in equations (3) and (3)', we obtain the optimal strategies:

$$p_1 = \sqrt{2} - 1,\ p_2 = \frac{1}{2},\ p_3 = 2 - \sqrt{2}\ ;$$
$$r_1 = \sqrt{2} - 1,\ r_2 = 1 - \frac{\sqrt{2}}{2},\ r_3 = \sqrt{2} - 1\ .$$

Summarizing, I's unique optimal strategy is (.414,.5,.585). II's unique optimal strategy is (.414,.293,.414). The value of the game (to I) is (.293,.5,.707). Thus, if I has 1 unit he has probability .293 of survival, etc.

The same play occurs for II's survival, with complementary probabilities of survival. This is so because the optimal strategies must obviously be mixed, so that there is zero probability of both players surviving. Hence, I may equivalently play for his own survival. This feature is true of any game with no zero entry:

$$a_{11},\ a_{22} > 0;\ a_{12},\ a_{21} < 0.$$

The case

$$A_0 = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$$

which was given before, illustrates the point that $f(p_1, \ldots , p_{n-1})$ mentioned before is not necessarily continuous. The play for I's survival is as expected (we take $n = 2$): $v(1) = 1$, $p_1 = 1$, $r_1 = 0$. But in the play for II's survival we consider the matrix

$$B_0 = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$$

and interchange the players. We have, using (4), (3) and (3)',

$v(1) = 0$, $p_1 = 0$, $r_1 = 1$. If this strategy is actually used, player I survives. Where is the contradiction? It lies in the continuity of f. For suppose player II (in the game B_0) announces a strategy of $(1 - \varepsilon, \varepsilon)$, ε small and positive. Player I either may use the strategy $(1, 0)$ in which case he survives with probability ε, or the strategy $(0, 1)$ in which case (because the game continues indefinitely) he survives with probability 0. But if $\varepsilon = 0$, I survives with probability 1 by choosing row 2. Hence, II may bring I's probability of survival as close to zero as he wishes without the possibility of reaching zero. In an actual play, ε would be taken very small and II would almost guarantee I's ruin.

In view of the above remarks, we may state the following theorem:

<u>Theorem</u>: Let $a_{11}, a_{22} > 0$, $a_{12}, a_{21} < 0$, and let n be arbitrary. Then the game has a solution. The plays for I's survival and for II's ruin are equivalent, and vice-versa.

mhb